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LETTER TO THE EDITOR

Infinitely many symmetries of the Davey–Stewartson equation

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Abstract. Using a formal series symmetry method for the Davey–Stewartson equation (DSE), we find that there exist two sets of infinitely many formal series symmetries of the DSE. However, different from the Kadomtsev–Petviashvili equation (KPE), we failed to get infinitely many truncated symmetries. Only six truncated symmetries can be obtained simply from the formal series symmetries.

The connection between the integrability properties of differential equations and their infinitely many symmetries is well known. An extensive literature on this subject already exists [1–4]. Given a system of differential equations, the problem of finding the infinitely many symmetries is often quite difficult. For $1+1$ dimensional cases, one of the effective methods is acting a recursion operator on a seed to get an infinite set of symmetries. However, the corresponding theory is much more complicated [5] for $2+1$ dimensional models. Recently, one of us (Lou) has developed a simple direct method to get the generalized symmetries for some $2+1$ dimensional equations [6, 7]. In [6], a set of generalized symmetries of the KPE $u_{tx} = (6uu_x - u_{xxx})_x - 3u_{yy}$ was expressed by a simple formula

$$K_n(f) = \frac{1}{2n!3^{n+1}} \sum_{k=0}^{n-1} f^{(n+1-k)}(-\partial_x^3 + 6\partial_x u - 3\partial_x^{-1}\partial_y^2 - \partial_t)^k y^n \qquad (n = 0, 1, 2, \ldots) \quad (1)$$

where f is an arbitrary function, $f^{(k)} = \frac{\partial^k}{\partial t^k} f$. These symmetries constitute a generalized w_∞ algebra

$$\begin{aligned}[K_n(f_1), K_m(f_2)] &\equiv \frac{\partial}{\partial \epsilon}[K_n(u + \epsilon K_m) - K_m(u + \epsilon K_n)] \\ &= \tfrac{1}{3} K_{m+n-2}((m+1)\dot{f}_1 f_2 - (n+1)\dot{f}_2 f_1) \end{aligned} \qquad (2)$$

with $\dot{f} = \frac{\partial}{\partial t} f$. The similar symmetry algebra structures are also found for the $2+1$ dimensional integrable dispersive long wave equation (IDLWE) [7], Nizhnik–Novikov–Veselov equation (NNVE) [8] and three-dimensional Toda field equation (TFE) [9]. Naturally, an important question now is whether all the $2+1$ dimensional integrable models have the

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generalized w_∞ symmetry algebra like (2). Unfortunately, in this letter we will give a negative answer.

It is known that the Davey–Stewartson equations (DSE) [10, 11]

$$
\begin{aligned}
&i\hbar\partial_t\psi + \frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)\psi - \kappa\psi^2\psi^* - \mu\psi\partial_y\phi = 0\\
&-i\hbar\partial_t\psi^* + \frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)\psi^* - \kappa\psi\psi^{*2} - \mu\psi^*\partial_y\phi = 0\\
&(\partial_x^2-\partial_y^2)\phi + \mu\partial_y(\psi\psi^*) = 0
\end{aligned}
\qquad (3)
$$

can be solved by the inverse scattering method both in the classical level [12] and in the quantum level [13, 14]. Champagne and Winternitz [15] have shown, using the classical Lie method, that the DS equation has an infinite-dimensional symmetry transformation group. The same conclusion can also be obtained as the gauge generalization of the symmetry transformation for the Schrödinger equation [11]. In this letter, we would like to use a very simple method to study the generalized symmetries of the DSE.

A symmetry of the DSE is defined as a solution of the linearized equation of (3):

$$
\begin{aligned}
&i\hbar\partial_t p + \frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)p - 2\kappa\psi\psi^* p - \kappa\psi^2 q - \mu p\partial_y\phi - \mu\psi\partial_y r = 0\\
&-i\hbar\partial_t q + \frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)q - 2\kappa\psi\psi^* q - \kappa\psi^{*2}p - \mu q\partial_y\phi - \mu\psi^*\partial_y r = 0\\
&(\partial_x^2-\partial_y^2)r + \mu\partial_y(p\psi^*) + \mu\partial_y(\psi q) = 0.
\end{aligned}
\qquad (4)
$$

In other words, DSE (3) is form-invariant under the infinitesimal transformation

$$
(\psi,\psi^*,\phi)^{\mathrm{T}} \to (\psi,\psi^*,\phi)^{\mathrm{T}} + \epsilon(p,q,r)^{\mathrm{T}} \qquad (5)
$$

where the superscript T means the transposition of matrix and ϵ is an infinitesimal parameter.

Now we see the solutions of (4) having a formal series form

$$
p_n = \sum_{k=0}^{\infty} f^{(n-1-k)}P_n[k] \qquad q_n = \sum_{k=0}^{\infty} f^{(n-1-k)}Q_n[k] \qquad r_n = \sum_{k=0}^{\infty} f^{(n-k)}R_n[k] \qquad (6)
$$

where f is an arbitrary function of t and $P_n[k]$, $Q_n[k]$, $R_n[k]$ are time-independent functions which should be determined later. It is conceivable that there are some other types of formal series symmetries, say f in (6) being an arbitrary function with two or more independent variables. However, it is quite difficult to get an explicit expression of a formal series symmetry except when the series has the form (6) and we failed to get any significant truncated symmetries other than those obtained from (6) (see below).

Substituting (6) into (4) yields

$$
\begin{aligned}
i\hbar\sum_{k=0}^{\infty} f^{(n-k)}P_n[k] + \sum_{k=0}^{\infty} f^{(n-1-k)}\Big\{ & i\hbar P_{nt}[k] + \frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)P_n[k] - 2\kappa\psi\psi^* P_n[k]\\
& - \kappa\psi^2 Q_n[k] - \mu P_n[k]\partial_y\phi\Big\} - \mu\sum_{k=0}^{\infty} f^{(n-k)}\psi\partial_y R_n[k] = 0
\end{aligned}
$$

$$-\mathrm{i}\hbar\sum_{k=0}^{\infty} f^{(n-k)}Q_n[k]+\sum_{k=0}^{\infty} f^{(n-1-k)}\Big\{-\mathrm{i}\hbar Q_{nt}[k]+\frac{\hbar^2}{2m}(\partial_x^2+\partial_y^2)Q_n[k]-2\kappa\psi\psi^* Q_n[k]$$
$$-\kappa\psi^{*2}P_n[k]-\mu Q_n[k]\partial_y\phi\Big\}-\mu\sum_{k=0}^{\infty} f^{(n-k)}\psi^*\partial_y R_n[k]=0$$
$$\sum_{k=0}^{\infty} f^{(n-k)}(\partial_x^2-\partial_y^2)R_n[k]+\mu\sum_{k=0}^{\infty} f^{(n-1-k)}\partial_y(P_n[k]\psi^*)+\mu\sum_{k=0}^{\infty} f^{(n-1-k)}\partial_y(\psi Q_n[k])=0. \tag{7}$$

Because f is an arbitrary function of t, equation (7) should be true at any order of time derivative (or time integration) of f. Then one can easily get all the undetermined functions, $P_n[k]$, $Q_n[k]$ and $R_n[k]$ from (7):

$$\partial_\xi\partial_\eta R_n[0]=0$$

i.e.

$$R_n[0]=g_n(\xi)+h_n(\eta) \qquad (\xi=x+y,\eta=x-y) \tag{8}$$
$$P_n[0]=Q_n^*[0]=\frac{\mu}{\mathrm{i}\hbar}\psi(R_{n\xi}[0]-R_{n\eta}[0]) \tag{9}$$
$$\begin{pmatrix}P_n[k]\\Q_n[k]\\R_n[k]\end{pmatrix}=\begin{pmatrix}-\partial_t+A & B^* & 0\\ B & -\partial_t+A^* & 0\\ C & C^* & 0\end{pmatrix}\begin{pmatrix}P_n[k-1]\\Q_n[k-1]\\R_n[k-1]\end{pmatrix}\equiv L\begin{pmatrix}P_n[k-1]\\Q_n[k-1]\\R_n[k-1]\end{pmatrix}$$
$$=L^2\begin{pmatrix}P_n[k-2]\\Q_n[k-2]\\R_n[k-2]\end{pmatrix}=\cdots=L^k\begin{pmatrix}P_n[0]\\Q_n[0]\\R_n[0]\end{pmatrix} \tag{10}$$

where $g_n(\xi)$ and $h_n(\eta)$ are two arbitrary functions of the indicated variables and the operators A, B and C are defined as

$$A\equiv\frac{\mathrm{i}\hbar}{m}(\partial_\xi^2+\partial_\eta^2)-\frac{2\mathrm{i}\kappa}{\hbar}\psi\psi^*-\frac{\mathrm{i}\mu}{\hbar}(\phi_\xi-\phi_\eta)-\frac{\mathrm{i}\mu^2}{4\hbar}\psi(2-\partial_\xi^{-1}\partial_\eta-\partial_\eta^{-1}\partial_\xi)\psi^*$$
$$B\equiv\frac{\mathrm{i}\kappa}{\hbar}\psi^{*2}+\frac{\mathrm{i}\mu^2}{4\hbar}\psi^*(2-\partial_\xi^{-1}\partial_\eta-\partial_\eta^{-1}\partial_\xi)\psi^*$$
$$C\equiv\frac{\mu}{4}(\partial_\xi^{-1}-\partial_\eta^{-1})\psi^* \qquad (\partial_\xi\partial_\xi^{-1}=\partial_\eta\partial_\eta^{-1}=1). \tag{11}$$

Because the symmetry definition equation (4) is linear and an arbitrary function can be expanded as a Laurent series, we can take the arbitrary functions $g_n(\xi)$ and $h_n(\eta)$ simply as $g_n(\xi)=\xi^n$, $h_n(\eta)=\eta^n$, $n=0,\pm1,\pm2,\ldots$. Finally, two sets of infinitely many formal series symmetries of the DSE are obtained. The first set of symmetries $\sigma_n^{(1)}(f)=(p_n(f),q_n(f),r_n(f))^{\mathrm{T}}$ is given by (6) with (8)–(11) and $g_n(\xi)=\xi^n$, $h_n(\eta)=0$. The other set of symmetries $\sigma_n^{(2)}(f)=(p_n(f),q_n(f),r_n(f))^{\mathrm{T}}$ is given by (6) with (8)–(11) and $g_n(\xi)=0$, $h_n(\eta)=\eta^n$.

For some other $2+1$ dimensional integrable models like the KPE, NNVE, IDLWE and TFE [6–9], we find that there exist one or more sets of infinitely many truncated symmetries.

However, after finishing some concrete calculation, we find only six of these two sets of formal series symmetries are truncated. The result reads

$$\begin{aligned}
&\sigma_0(f) \equiv \sigma_0^{(1)}(f) = \sigma_0^{(2)}(f) = \frac{mf}{\mu}\begin{pmatrix}0\\0\\1\end{pmatrix}\\
&\sigma_1^{(1)}(f) = \frac{m}{\hbar\mu}\begin{pmatrix}-\mathrm{i}\mu f\psi\\ \mathrm{i}\mu f\psi^*\\ \hbar f\xi\end{pmatrix} \qquad \sigma_1^{(2)}(f) = \frac{m}{\hbar\mu}\begin{pmatrix}\mathrm{i}\mu f\psi\\ -\mathrm{i}\mu f\psi^*\\ \hbar f\eta\end{pmatrix}
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\sigma_2^{(1)}(f) = \frac{1}{4\hbar\mu}\begin{pmatrix}4\hbar\mu f\psi_\xi - 2\mathrm{i}m\mu\xi\dot f\psi\\ 4\hbar\mu f\psi_\xi^* + 2\mathrm{i}m\mu\xi\dot f\psi^*\\ 4\hbar\mu f\phi_\xi + m\hbar\xi^2\ddot f\end{pmatrix}\\
&\sigma_2^{(2)}(f) = \frac{1}{4\hbar\mu}\begin{pmatrix}4\hbar\mu f\psi_\eta - 2\mathrm{i}m\mu\eta\dot f\psi\\ 4\hbar\mu f\psi_\eta^* + 2\mathrm{i}m\mu\eta\dot f\psi^*\\ 4\hbar\mu f\phi_\eta + m\hbar\eta^2\ddot f\end{pmatrix}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\sigma_3(f) &\equiv \sigma_3^{(1)}(f) - \sigma_3^{(2)}(f)\\
&= \frac{1}{24\hbar\mu}\begin{pmatrix}24\hbar\mu f\psi_t + 2\hbar\mu\dot f(x\psi_x + y\psi_y + \psi) - 3\mathrm{i}m\mu(\xi^2+\eta^2)\ddot f\psi\\ 24\hbar\mu f\psi_t^* + 2\hbar\mu\dot f(x\psi_x^* + y\psi_y^* + \psi^*) + 3\mathrm{i}m\mu(\xi^2+\eta^2)\ddot f\psi^*\\ 24\hbar\mu f\phi_t + 2\hbar\mu\dot f(x\phi_x + y\phi_y + \phi) + m\hbar(\xi^3-\eta^3)\dddot f\end{pmatrix}
\end{aligned} \tag{14}$$

while others of these two sets of formal series symmetries cannot be truncated. It is interesting that six truncated symmetries given by (12)–(14) constitute an infinite-dimensional Lie algebra under the Lie product

$$[A, B] = \frac{\partial}{\partial\epsilon}[A(u+\epsilon B) - B(u+\epsilon A)]|_{\epsilon=0} \equiv A'B - B'A \tag{15}$$

for $A = (a_1, a_2, a_3)^{\mathrm{T}}$, $B = (b_1, b_2, b_3)^{\mathrm{T}}$ and $u = (\psi, \psi^*, \phi)^{\mathrm{T}}$. The non-trivial commutation relations read ($i = 1, 2$):

$$[\sigma_0(f_1), \sigma_3(f_2)] = [\sigma_1^{(i)}(f_2), \sigma_2^{(i)}(f_1)] = -\sigma_0(f_1\dot f_2) \tag{16}$$

$$[\sigma_1^{(i)}(f_1), \sigma_3(f_2)] = -\sigma_1^{(i)}(f_2\dot f_1) \qquad [\sigma_2^{(1)}(f_1), \sigma_2^{(1)}(f_2)] = \tfrac{1}{2}\sigma_1^{(1)}(f_1\dot f_2 - f_2\dot f_1) \tag{17}$$

$$[\sigma_2^{(2)}(f_1), \sigma_2^{(2)}(f_2)] = -\tfrac{1}{2}\sigma_1^{(2)}(f_1\dot f_2 - f_2\dot f_1) \qquad [\sigma_2^{(i)}(f_1), \sigma_3(f_2)] = \tfrac{1}{2}\sigma_2^{(i)}(f_1\dot f_2 - 2f_2\dot f_1) \tag{18}$$

$$[\sigma_3(f_1), \sigma_3(f_2)] = \sigma_3(f_1\dot f_2 - f_2\dot f_1). \tag{19}$$

From (16)–(18) and (19), we see that $\{\sigma_0(f), \sigma_1^{(i)}(f), \sigma_2^{(i)}(f)\} \equiv L$ and $\{\sigma_3(f)\} \equiv S$ constitute two subalgebras of (16)–(19). Furthermore, there exist some types of interesting subalgebras of (19). For instance:

(i) If we take $m = n = 2$, $f = \exp(rt/\alpha)(r = 0, \pm1, \pm2, \ldots)$, $\alpha =$ constant, we get the first type of the Virasoro algebra from (19), $\sigma^r \equiv \sigma_3(\exp rt/\alpha)$:

$$[\sigma^r, \sigma^s] = \frac{1}{\alpha}(r-s)\sigma^{r+s} \qquad (r, s = 0, \pm1, \pm2, \ldots). \tag{20}$$

(ii) When we take $m = n = 2$, $f = (1/\alpha)t^r$, $(r = 0, \pm 1, \pm 2, \ldots)$, $\alpha =$ constant, the algebra (19) becomes the second type of Virasoro algebra, $\sigma^r \equiv \sigma_3(t^r/\alpha)$:

$$[\sigma^r, \sigma^s] = \frac{1}{\alpha}(r - s)\sigma^{r+s-1} \qquad (r, s = 0, \pm 1, \pm 2, \ldots). \tag{21}$$

(iii) If we take $\sigma_3(f)$ possessing the form $(\sigma_i^r \equiv \sigma_3(t^{r/p}\exp it/\alpha)$, $(r, i = 0, \pm 1, \pm 2, \ldots)$, $\alpha =$ constant, $p \in Z$, then we get a w_∞-type algebra [16]:

$$[\sigma_i^r, \sigma_j^s] = \frac{1}{\alpha p}(r - s)\sigma_{i+j}^{r+s-p} + \frac{1}{\alpha}(i - j)\sigma_{i+j}^{r+s}. \tag{22}$$

The algebra (22) will reduce back to the Virasoro algebras (20) and (21) for $(r = s = 0)$ and $(i = j = 0, p = 1)$, respectively.

Using much more difficult approaches, the truncated symmetries (12)–(14) and its algebra (16)–(19) are also obtained in [15, 11] where some different but equivalent notations are used.

In summary, there exist two sets of infinitely many formal series symmetries for the DSE. However, different from the KPE, IDLWE, NNVE and the TFE, only six truncated symmetries can be found. After extending the same method to some other $2+1$ integrable models, such as the $2+1$ dimensional Sawada–Kotera equation, $2+1$ dimensional Ito equation and the coupled KdV–Ito equation [17], as in the DSE case, one can find that there exist one or two sets of infinitely many formal series symmetries, but only finite numbers of truncated symmetries can be found. Now, two important open questions arise: What is the relation between the integrability and the formal series symmetry truncated condition? And how many truncated symmetries are required to guarantee the integrability of a high-dimensional model? To answer the latter question, we have a theory which we aim to prove later: If there exists a truncated symmetry $\sigma(f)$ (say $\sigma_3(f)$ for the DSE) with arbitrary function $f(t)$ for a high-dimensional model such that $\sigma(f)$ satisfies the algebra (19), then the model may be integrable. Though we cannot prove this theory now, all the $2+1$ dimensional integrable models known before, say the models in [17], have such a symmetry subalgebra and we have not yet found a non-integrable model possessing such a type of symmetry algebra.

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